

NONNEGATIVE RANK DEPENDS ON THE FIELD II

YAROSLAV SHITOV

ABSTRACT. We provide an example of a 21×21 matrix with nonnegative integer entries which can be written as a sum of 19 nonnegative rank-one matrices but not as a sum of 19 rational nonnegative rank-one matrices. This gives a solution for a problem posed by Cohen and Rothblum in 1993.

Let A be a matrix with nonnegative entries in a field $\mathcal{F} \subset \mathbb{R}$. The nonnegative rank of A with respect to \mathcal{F} is the smallest k such that A is a sum of k nonnegative rank-one matrices with entries in \mathcal{F} . We denote this quantity by $\text{Rank}_+(A, \mathcal{F})$ or simply $\text{Rank}_+ A$ if $\mathcal{F} = \mathbb{R}$. Cohen and Rothblum asked the following question in the foundational paper [1] published in 1993.

Problem 1. Is there a rational matrix A such that $\text{Rank}_+(A, \mathbb{Q}) \neq \text{Rank}_+(A)$?

Applications motivating this problem include the theory of computation. Vavasis [6] demonstrates the connection between Problem 1 and algorithmic complexity of nonnegative rank. Another notable application of nonnegative ranks is the theory of extended formulations of polytopes [7], and the possible lack of optimal rational factorizations is a difficulty in this theory [4]. Kujas, Robeva, and Sturmfels [3] consider Problem 1 in context of modern statistics; they also give a partial solution of this problem. The work on relaxed versions includes the first part of this paper [5] with an example of a field \mathcal{F} and a matrix A over \mathcal{F} such that $\text{Rank}_+(A, \mathcal{F}) \neq \text{Rank}_+(A)$. A similar problem for *positive semidefinite* rank has been solved in [2].

The aim of this note is to give a positive solution of Problem 1 by proving the result mentioned in the abstract. Assuming that all variables are nonnegative, we denote by $\mathcal{B}(\alpha_1, \dots, \alpha_n)$ and $\mathcal{C}(a_1, a_2, b, c, d)$ the matrices

$$\begin{pmatrix} \alpha_1 & \dots & \alpha_n & 1 & 1 & 1 & 1 \\ 1 & \dots & 1 & 1 & 1 & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 & 1 \\ 0 & \dots & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} d & 2 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & b & 0 \\ 0 & 1 & 0 & 0 & c \\ 0 & 1 & 1 & a_1 & a_2 \end{pmatrix},$$

and by V the bottom-right 4×4 submatrix of \mathcal{B} . The lower bounds in Claims 2–4 can be easily checked for the conventional rank function, and it is sufficient for the proof as the inequality $\text{Rank} \leq \text{Rank}_+$ shows. Claim 5 is a basic example, see [1].

Claim 2. $\text{Rank}_+ \mathcal{C} \geq 3$.

2000 *Mathematics Subject Classification.* 15A23.

Key words and phrases. Nonnegative matrix factorization.

Proof. Let $\alpha = 1 + \sqrt{0.5}$. It is easy to check that the equalities $a_1 = b = c = \alpha$, $d = \sqrt{2}$ follow from a weaker statement $\text{Rank } \mathcal{C} \leq 3$. To prove the opposite direction, we check that the rows of $\mathcal{C}(\alpha, \alpha, \alpha, \alpha, \sqrt{2})$ are sums of the rows $(0, 1, 0, 0, \alpha)$, $(0, 0, 1, \alpha, 0)$, $(\sqrt{2}, 2, \sqrt{2}, 0, 0)$ multiplied by nonnegative coefficients. \square

[illegible]

Proof. Let $\alpha = 1 + \sqrt{0.5}$, $M_1 = \mathcal{C}(\alpha, \alpha, \alpha, \alpha, \sqrt{2})$, $M_2 = \mathcal{B}(2 - \alpha, 2 - \alpha)$, $M_3 = \mathcal{B}(2 - \alpha)$, $M_4 = \mathcal{B}(2 - \sqrt{2})$. We subtract M_1 from the green submatrix of \mathcal{A} , and we denote the resulting matrix by A . We note that the non-zero entries of A are covered by the disjoint submatrices $A(5, 6, 7, 8, 9|4, 5, 6, 7, 8, 9)$, $A(4, 10, 11, 12, 13|5, 10, 11, 12, 13)$, $A(3, 14, 15, 16, 17|4, 14, 15, 16, 17)$, $A(1, 18, 19, 20, 21|1, 18, 19, 20, 21)$. These matrices are M_2 , M_3 , M_3 , M_4 respectively, so we get $\text{Rank}_+ \mathcal{A} \leq \text{Rank}_+ M_1 + \text{Rank}_+ M_2 + 2 \text{Rank}_+ M_3 + \text{Rank}_+ M_4 = 19$ from Claims 6, 7. \square

Claim 9. $\text{Rank}_+(\mathcal{A}, \mathbb{Q}) \geq 20$.

Proof. Assuming the converse, we get $\mathcal{A} = A_1 + \dots + A_{19}$ where A_i 's are rational rank-one nonnegative matrices. We say that A_i is a **red** (or **blue**, **yellow**, **magenta**) *summand* if it has at least one non-zero entry with the corresponding color. Since all the entries with these colors are covered by the bottom-right 16×16 submatrix of \mathcal{A} , and since this submatrix has the form $\text{diag}(\mathbf{V}, \mathbf{V}, \mathbf{V}, \mathbf{V})$, we can apply Claim 5 and conclude that the colors do not intersect and contain at least four A_i 's each. We define $\mathbf{R}, \mathbf{B}, \mathbf{Y}, \mathbf{M}$ as the sums of all A_i 's that belong to the corresponding colors. We denote by U the sum of uncolored A_i 's, and we get $\text{Rank}_+ U \leq 3$.

Let us say that *main* entries are those colored **light blue** or **green**. We note that the possible non-zero main entries for **red summands** are the light blue entries and $(5, 4), (5, 5)$. For **yellow summands**, the only possible non-zero main entry is $(3, 4)$, for **blue summands** only $(4, 5)$ is possible, and for **magenta summands** only $(1, 1)$ is possible. This shows that the green submatrix of U has the form \mathcal{C} . Claim 2 implies $\text{Rank}_+ U = 3$, so we have $\text{Rank}_+ \mathbf{R} \leq 4$.

If U has a non-zero light blue entry, then it contains a submatrix as in Claim 3, which is a contradiction. Otherwise, the red and light blue entries of \mathbf{R} are equal to those of \mathcal{A} , which means that the submatrix $\mathbf{R}[5, 6, 7, 8, 9 | 4, 5, 6, 7, 8, 9]$ has the form \mathcal{B} . We see from Claim 4 that the $(5, 4)$ and $(5, 5)$ entries are equal in \mathbf{R} ; this shows that the $(5, 4)$ and $(5, 5)$ entries are equal in U as well. We apply Claim 7 and get a contradiction with the rationality of U . \square

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NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS, 20 MYASNITSKAYA ULITS, MOSCOW, RUSSIA 101000

E-mail address: yaroslav-shitov@yandex.ru